

# NUMERICAL COMPUTATIONS ON THE ZEROS OF THE EULER DOUBLE ZETA-FUNCTION II

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**ABSTRACT.** We study the behavior of zero-divisors of the double zeta-function  $\zeta_2(s_1, s_2)$ . In our former paper [4] we studied the case  $s_1 = s_2$ , but in the present paper we consider the more general two variable situation. We carry out numerical computations in order to trace the behavior of zero-divisors. We observe that some divisors approach the points with  $s_2 = 0$ , while other divisors approach the points which are solutions of  $\zeta(s_2) = 1$ .

## 1. INTRODUCTION

The present paper is a continuation of the authors' previous article [4] on the study of the zeros of the Euler double zeta-function

$$(1.1) \quad \zeta_2(s_1, s_2) = \sum_{n_1=1}^{\infty} \sum_{n_2=1}^{\infty} \frac{1}{n_1^{s_1} (n_1 + n_2)^{s_2}},$$

where  $s_1, s_2$  are complex variables. This double series is convergent absolutely in the region defined by  $\Re s_1 + \Re s_2 > 2$  and  $\Re s_2 > 1$ , and can be continued meromorphically to the whole complex space  $\mathbb{C}^2$  (see [3]).

This is the case  $r = 2$  of the more general Euler-Zagier  $r$ -ple sum

$$(1.2) \quad \zeta_r(s_1, \dots, s_r) = \sum_{1 \leq n_1 < \dots < n_r} \frac{1}{n_1^{s_1} \dots n_r^{s_r}},$$

which has been investigated quite extensively from various aspects. However, the distribution of the zeros of (1.2) has not been, except for the classical case of  $r = 1$  (that is the case of the Riemann zeta-function  $\zeta(s)$ ), studied in detail. In order to understand the analytic properties of (1.2), it is very important to study the behavior of its zeros. The aim of the present series of papers is to study the behavior of the zeros of (1.1), the simplest case (except for the case  $r = 1$ ), from the viewpoint of numerical computations.

In [4], we considered the situation when  $s_1 = s_2 (= s)$ . Then  $\zeta_2(s, s)$  is a function of one variable, so we can study the distribution of the

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zeros of  $\zeta_2(s, s)$  in a way analogous to the case of the Riemann zeta-function. Unlike the case of  $\zeta(s)$ , the function  $\zeta_2(s, s)$  does not satisfy the analogue of the Riemann hypothesis. We found a lot of zeros in the strip  $0 \leq \Re s \leq 1$  off the line  $\Re s = 1/2$ , or even outside that strip (see [4, Observation 1] and [4, Figure 1]). We pointed out that the distribution of those zeros is similar, not to that of  $\zeta(s)$ , but rather, to that of Hurwitz zeta-functions.

Inspired by [4], Ikeda and Matsuoka proved several theoretical results on the distribution of the zeros of  $\zeta_2(s, s)$  in [2].

In the present paper, we consider the general situation, when  $s_1$  and  $s_2$  are moving independently. Then  $\zeta_2(s_1, s_2)$  is a function of two variables, so its zeros are not isolated points, but they form zero-divisors. We carry out numerical computations in order to trace the behavior of such zero-divisors. Our basic strategy is to begin with the zeros of  $\zeta_2(s, s)$  discovered in [4], and study the behavior of zero-divisors of  $\zeta_2(s_1, s_2)$  around those zeros. Then we consider the asymptotic situation of those zero-divisors when, for example,  $|s_1 - s_2|$  becomes large, or  $s_2 \rightarrow 0$ , or  $|s_1| \rightarrow \infty$ . We observe that some divisors approach the points with  $s_2 = 0$ , while other divisors approach the points which are solutions of  $\zeta(s_2) = 1$ .

## 2. THE BEHAVIOR OF ZERO-DIVISORS

Let us begin with [4, Figure 1], on which a lot of non-real zeros of  $\zeta_2(s, s)$  are dotted. The values of some of which are given in the list written on [4, pp.308-309], whose order is according to the magnitude of the imaginary parts of them. Denote those zeros by  $a_1, a_2, a_3, \dots$  and so on.

Since  $\zeta_2(s_1, s_2)$  cannot have any isolated zero point, these  $a_i$ s are to be intersections of some zero-divisors and the hyperplane  $s_1 = s_2$ . Our first aim is to investigate the behavior of zero-divisors near the points  $a_i$ .

Let  $\delta$  be a positive number. We search for the zeros of  $\zeta_2(s_1, s_2)$  around the point  $a_i$  under the condition  $|s_1 - s_2| = \delta$ . The method of computations is based on the Euler-Maclaurin formula, explained in [4, Section 4]. (It is to be noted that the simple method using the harmonic product formula [4, (2.1), (2.2)] cannot be applied to the present situation where  $s_1 \neq s_2$ .) Figure 1 describes the loci of the absolute values of zeros satisfying  $|s_1 - s_2| = \delta$  around  $a_i$  ( $5 \leq i \leq 9$ )

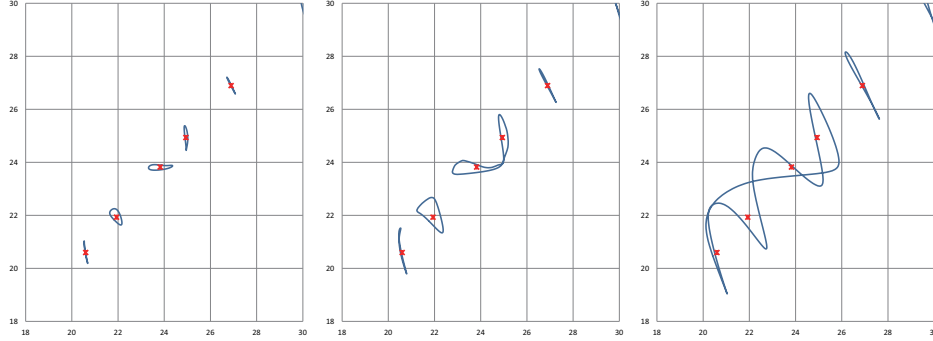


FIGURE 1. The loci of the absolute values of zeros of  $\zeta_2(s_1, s_2)$  around  $a_i$  ( $5 \leq i \leq 9$ ) with  $|s_1 - s_2| = \delta$ , where  $\delta = 0.5$  (left),  $1.0$  (center),  $2.0$  (right). The horizontal axis represents  $|s_1|$  and the vertical axis does  $|s_2|$ . The points indicated by the cross marks are  $|a_5|, |a_6|, |a_7|, |a_8|, |a_9|$  from the lower-left to the upper-right.

for various values of  $\delta$ . We quote the values of those  $a_i$ s from [4]:

$$\begin{aligned} a_5 &= (0.12844956) + i(20.59707674), \\ a_6 &= (0.08804454) + i(21.93232180), \\ a_7 &= (1.10778631) + i(23.79708697), \\ a_8 &= (0.27268471) + i(24.93425087), \\ a_9 &= (-0.67413685) + i(26.88584448). \end{aligned}$$

The left one of Figure 1 is the situation when  $\delta = 0.5$ . In this figure we can observe that zero-divisors around  $a_7$  and  $a_8$  become closer to each other, and it seems that these two zero-divisors are connected in the central figure, when  $\delta = 1$ . In the right figure, all the zero-divisors around  $a_5$  to  $a_8$  seem connected. When  $\delta$  becomes larger, more and more zero-divisors seem to be connected with each other (see Figure 2).

Since Figure 1 only represents the behavior of absolute values, in order to make sure that the above zero-divisors are indeed connected, it is necessary to investigate the values of real parts and imaginary parts of those. Figure 3 gives such data. This figure shows that the loci of zeros around  $a_7$  and  $a_8$  are indeed connected.

From Figure 2 it seems that all zero-divisors appearing in this figure are connected. (The divisors including the points  $a_1$  and  $a_2$  are not

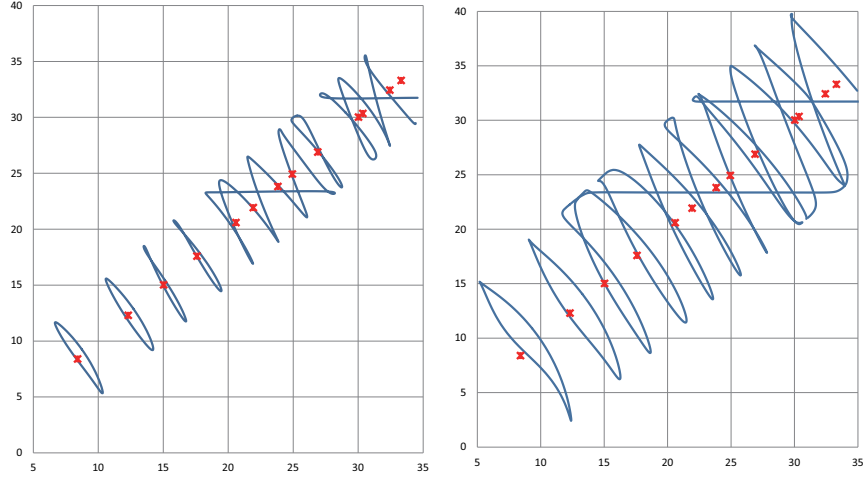


FIGURE 2. The loci of the absolute values of zeros of  $\zeta_2(s_1, s_2)$  around  $a_i$  ( $1 \leq i \leq 13$ ) with  $|s_1 - s_2| = \delta$ , where  $\delta = 5.0$  (left), and  $10.0$  (right).

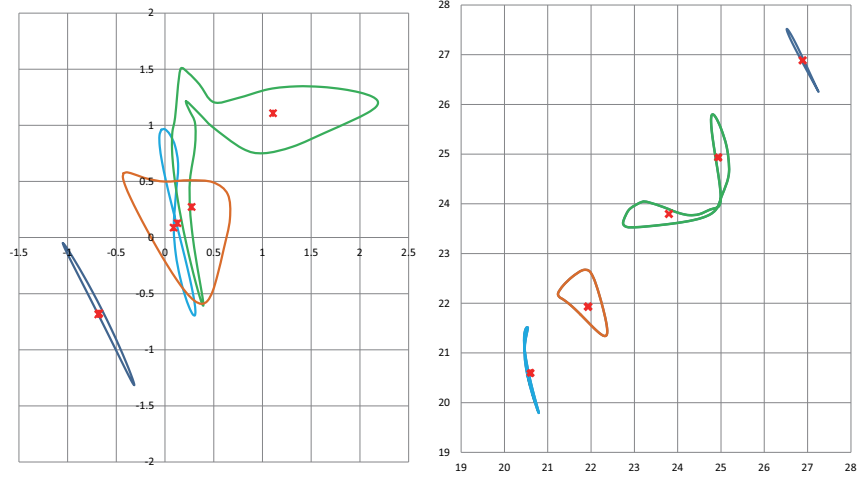


FIGURE 3. The behavior of real parts and imaginary parts of zeros around  $a_i$  ( $5 \leq i \leq 9$ ) when  $\delta = 1.0$ . On the left figure, the horizontal axis represents  $\Re s_1$ , and the vertical axis does  $\Re s_2$ , while on the right figure they represent  $\Im s_1$  and  $\Im s_2$ . Since the absolute value is almost determined by the imaginary part (because the real part is relatively small), the right figure is very similar to the central one of Figure 1.

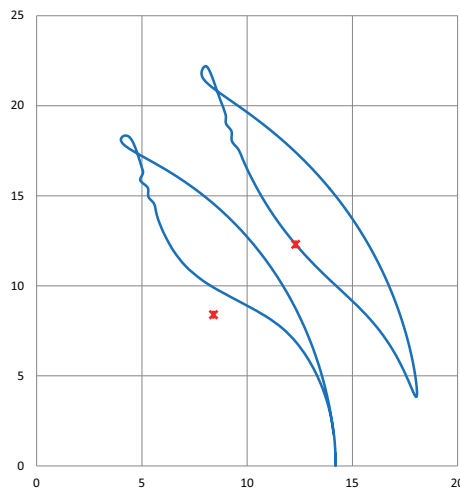


FIGURE 4. The loci of the absolute values of zeros of  $\zeta_2(s_1, s_2)$  around  $a_1$  and  $a_2$  with  $|s_1 - s_2| = \delta$ , where  $\delta = 14.0$ .

connected to the other divisors on the figure, but further computations show that these divisors look connected also, when  $\delta$  becomes larger.)

From this observation, perhaps we may expect that all  $a_1, a_2, a_3, \dots$  are lying on the same (unique?) zero-divisor. However, later in Section 4 we will see that the behavior of some zero-divisors is rather different. Probably it is too early to raise any conjecture on the behavior of zero-divisors.

*Remark 1.* In [4, p.308], it is noted that the left-most zero in [4, Figure 1] is

$$(-0.830372) + i(35.603804).$$

However, there is at least one zero of  $\zeta_2(s, s)$  which is located more left. When the authors wrote [4], they overlooked the following two zeros:

$$\begin{aligned} &(-0.874058504) + i(44.93750365), \\ &(-0.710036436) + i(53.91464901). \end{aligned}$$

### 3. APPROACHING THE AXIS $s_2 = 0$

From Figure 2 we can observe that, when  $\delta$  becomes larger, the curves consisting of zeros also becomes larger, and the bottoms of the curves look approaching to the horizontal axis (that is, the axis  $s_2 = 0$ ). Figure 4 describes the curves of the absolute values of zeros when  $\delta = 14.0$ . In this case one curve indeed touches the horizontal axis.

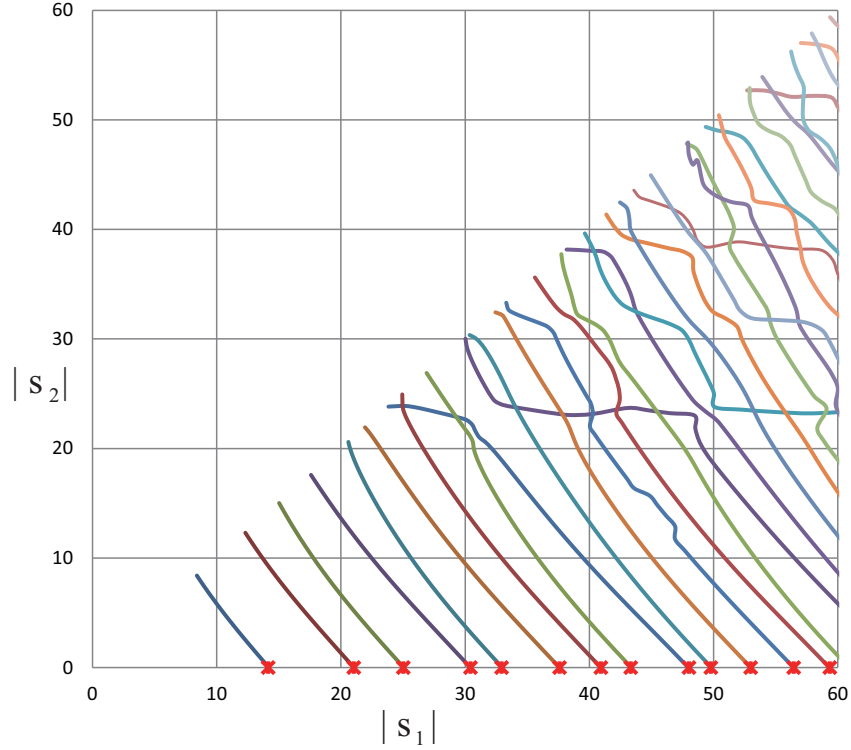


FIGURE 5. The absolute values of zeros satisfying  $s_2 = \eta s_1$ , where  $1 \geq \eta \geq 0$ . The horizontal axis represents  $|s_1|$  (up to 60) and the vertical axis does  $|s_2|$ . The cross marks represent the absolute values of zeros of  $\zeta(s)$ .

How about the behavior of other curves? To investigate this point, now we use an alternative way of calculating zeros. Consider the equation  $s_2 = \eta s_1$ . In [4], we studied the zeros under the condition  $\eta = 1$ . Starting with the data of those zeros, we search for the zeros for various values of  $\eta$ ,  $1 \geq \eta \geq 0$ . When  $\eta \rightarrow 0$ , we find that almost all curves consisting of zeros tend to the horizontal axis (see Figure 5).

From Figure 5 we observe that the points at which the loci touch the horizontal axis are almost the same as the absolute values of zeros of  $\zeta(s)$ . Let us list up those values. The left of Table 1 is the list of the values of  $s_1$  of the points where the curves touch the horizontal axis. Comparing this table with the list of non-trivial zeros of  $\zeta(s)$  (the right of Table 1), we find:

TABLE 1. The left is the list of the values of  $s_1$  of the points where the curves touch the horizontal axis. The right is the list of non-trivial zeros of  $\zeta(s)$ .

$(1.247595281) + i(14.14857043)$	$(0.5) + i(14.13472514)$
$(1.279113136) + i(21.01244258)$	$(0.5) + i(21.02203964)$
$(1.292752716) + i(25.03054326)$	$(0.5) + i(25.01085758)$
$(1.304538379) + i(30.39099998)$	$(0.5) + i(30.42487613)$
$(1.310484619) + i(32.97276761)$	$(0.5) + i(32.93506159)$
$(1.319356152) + i(37.57182573)$	$(0.5) + i(37.58617816)$
$(1.320370441) + i(40.90318345)$	$(0.5) + i(40.91871901)$
$(1.328328378) + i(43.36573059)$	$(0.5) + i(43.32707328)$
$(1.333209526) + i(47.94829355)$	$(0.5) + i(48.00515088)$
$(1.330151768) + i(49.81347595)$	$(0.5) + i(49.77383248)$
$(1.338852528) + i(52.98250152)$	$(0.5) + i(52.97032148)$
$(1.341784096) + i(56.42981942)$	$(0.5) + i(56.44624769)$
$(1.336083544) + i(59.30477438)$	$(0.5) + i(59.34704400)$
$(1.347599692) + i(60.90235918)$	$(0.5) + i(60.83177852)$
$(1.350310992) + i(65.06828349)$	$(0.5) + i(65.11254405)$
$(1.343496503) + i(67.09040611)$	$(0.5) + i(67.07981053)$
$(1.348806781) + i(69.55741541)$	$(0.5) + i(69.54640171)$
$(1.354734356) + i(72.08670993)$	$(0.5) + i(72.06715767)$
$(1.356705834) + i(75.63889778)$	$(0.5) + i(75.70469069)$
$(1.344904017) + i(77.17673184)$	$(0.5) + i(77.14484007)$
$(1.360065509) + i(79.37508073)$	$(0.5) + i(79.33737502)$
$(1.360772059) + i(82.86977267)$	$(0.5) + i(82.91038085)$
$(1.355304386) + i(84.75005063)$	$(0.5) + i(84.73549298)$
$(1.353373038) + i(87.38999143)$	$(0.5) + i(87.42527461)$
$(1.365347698) + i(88.87469754)$	$(0.5) + i(88.80911121)$
$(1.367563235) + i(92.45429247)$	$(0.5) + i(92.49189927)$
$(1.352878941) + i(94.60615786)$	$(0.5) + i(94.65134404)$
$(1.364213735) + i(95.93955674)$	$(0.5) + i(95.87063423)$
$(1.367791276) + i(98.82956582)$	$(0.5) + i(98.83119422)$

**Observation 1.** *The imaginary part of each  $s_1$  in the left list of Table 1 is very close to an imaginary part of a value appearing in the right list of Table 1.*

This phenomenon can be (heuristically) explained as follows. Recall the formula ([4, (2.3)], originally in [1]):

$$(3.1) \quad \zeta_2(s_1, s_2) = \frac{\zeta(s_1 + s_2 - 1)}{s_2 - 1} - \frac{\zeta(s_1 + s_2)}{2} \\ + \sum_{q=1}^l (s_2)_q \frac{B_{q+1}}{(q+1)!} \zeta(s_1 + s_2 + q) - \sum_{n_1=1}^{\infty} \frac{\phi_l(n_1, s_2)}{n_1^{s_1}},$$

where  $(s_2)_q = s_2(s_2+1) \cdots (s_2+q-1)$ ,  $B_{q+1}$  is the  $(q+1)$ -th Bernoulli number, and  $\phi_l(n_1, s_2)$  is as [4, (2.4)]. Put  $s_2 = 0$  in (3.1). Since  $\phi_l(n_1, 0) = 0$  (which can be seen by [4, (2.5)]), we find that the two sums on the right-hand side of (3.1) are both zero, so

$$(3.2) \quad \zeta_2(s_1, 0) = -\zeta(s_1 - 1) - \frac{\zeta(s_1)}{2}.$$

This implies that  $s_1$  in the left list of Table 1 satisfies

$$(3.3) \quad \zeta(s_1) = -2\zeta(s_1 - 1).$$

Let

$$C(t) = \{\zeta(\sigma + it) \mid \sigma \in \mathbb{R}\}.$$

Then as can be seen from Figure 6, the slope of the graph of the curve  $C(t)$  does not so rapidly change. This can be naturally expected, because the approximate functional equation of  $\zeta(s)$  (see [5, Theorem 4.15]) implies that the behavior of  $\zeta(s)$  is dominated by the terms of the form  $n^{-s} = n^{-\sigma} e^{-it \log n}$ , and if  $t$  is fixed, then the “argument” part of these terms does not change.

Therefore, for each  $t$ , we can find a number  $\sigma(t) \in \mathbb{R}$  such that

$$(3.4) \quad |\zeta(s(t))| = 2|\zeta(s(t) - 1)|,$$

where  $s(t) = \sigma(t) + it$ . We denote by  $L(t)$  the segment joining  $\zeta(s(t))$  and  $\zeta(s(t) - 1)$ .

When  $(1/2) + it_0$  is in the right list of Table 1, as in the upper one of Figure 6 (where the case  $t_0 = 14.13472514$  is described),  $\arg \zeta(s(t_0) - 1)$  is almost equal to  $\arg \zeta(s(t_0)) \pm \pi$ , so  $L(t_0)$  passes very close to the origin. (If  $C(t_0)$  would be a straight line, then  $L(t_0)$  could indeed cross the origin; but this is not the case.) Move the value of  $t$  a little from  $t_0$ . Then the curve  $C(t)$  also moves a little. Then, as in the lower one of Figure 6, we may find a value of  $t = t_0^*$ , close to  $t_0$ , for which  $L(t_0^*)$  indeed crosses the origin. This implies  $\zeta(s(t_0^*)) = -2\zeta(s(t_0^*) - 1)$ , that is, in view of (3.3), this  $s(t_0^*) = \sigma(t_0^*) + it_0^*$  should be in the left list of Table 1.



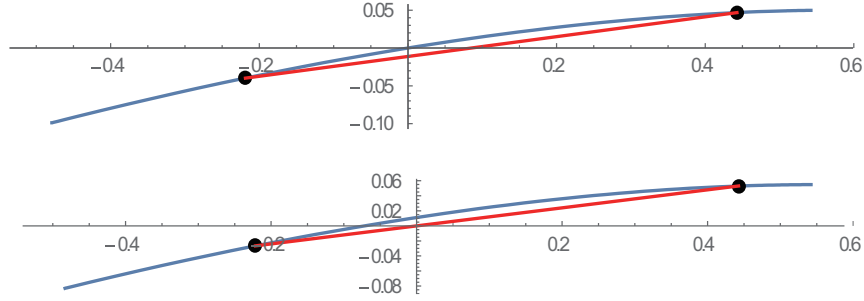


FIGURE 6. The first one is the locus of  $\zeta(\sigma + it_0)$ , where  $t_0 = 14.13472514$  (= the first of the right list of Table 1). Therefore this locus crosses the origin. The two black-dots represent the points  $s(t_0) = (0.442629) + i(0.0469269)$  and  $s(t_0 - 1) = (-0.218684) - i(0.0397677)$ , with  $\sigma(t_0) = 1.2475$ . This satisfies (3.4). The second one is the locus of  $\zeta(\sigma + it_0^*)$ , where  $t_0^* = 14.14857043$  (= the first of the left list of Table 1).

*Remark 2.* It is also observed that the real parts of the points on the list of Table 1 are close to each other, around the value 1.3 (which can be also seen from Figure 9).

#### 4. APPROACHING THE ZEROS OF $\zeta(s) = 1$

In Figure 5, we can observe that almost all curves approach the horizontal axis. However, when we extend the range of computations, we find that there are curves which do not seem to approach the horizontal axis (Figure 7). Along these curves, it seems that  $|s_1|$  becomes larger and larger.

Moreover, extending the range of computations, we can find more divisors, along them  $|s_1|$  seems to tend to infinity (see Figure 8).

The numerical data suggests that  $\Re s_1$  tends to infinity along these curves (Figure 9 and the left of Figure 10), while  $\Re s_2$  remains finite (the right of Figure 10).

What happens? The following proposition is due to Professor Seidai Yasuda (Osaka University). The authors express their sincere gratitude to him for the permission of including the following result in the present paper.

**Proposition 1.** (i) Let  $(s_1^{(m)}, s_2^{(m)})$  ( $m = 1, 2, \dots$ ) be a sequence of points on a zero-divisor of  $\zeta_2(s_1, s_2)$ . If  $\sigma_1^{(m)} = \Re s_1^{(m)}$  tends to infinity

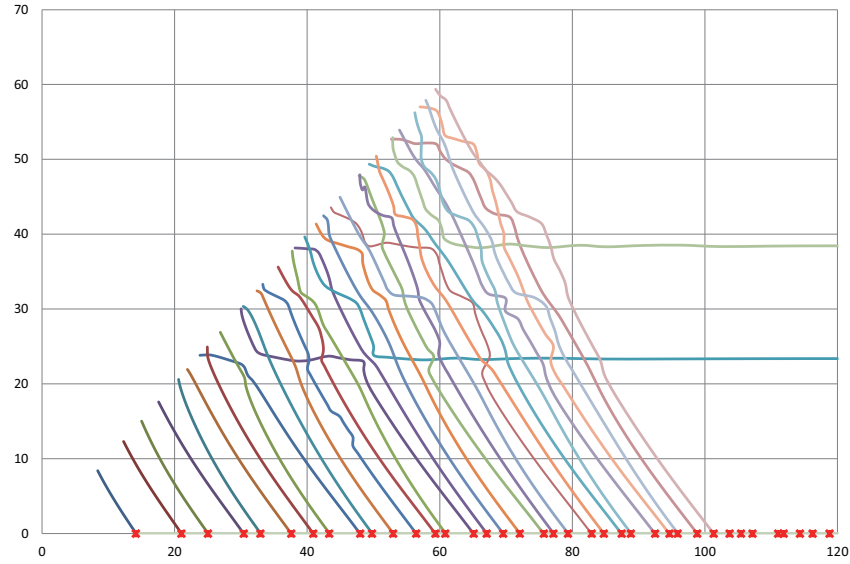


FIGURE 7. This figure represents the same computations as in Figure 5, but the range of  $|s_1|$  is up to 120. The cross marks represent the absolute values of zeros of  $\zeta(s)$ . There appear two curves which do not approach the horizontal axis.

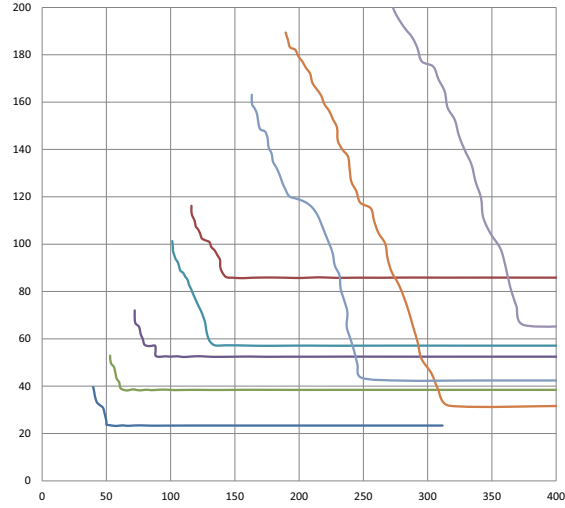


FIGURE 8. The same computations as in Figure 5, but the range of  $|s_1|$  is up to 400. Only the curves which do not approach the horizontal axis are drawn.

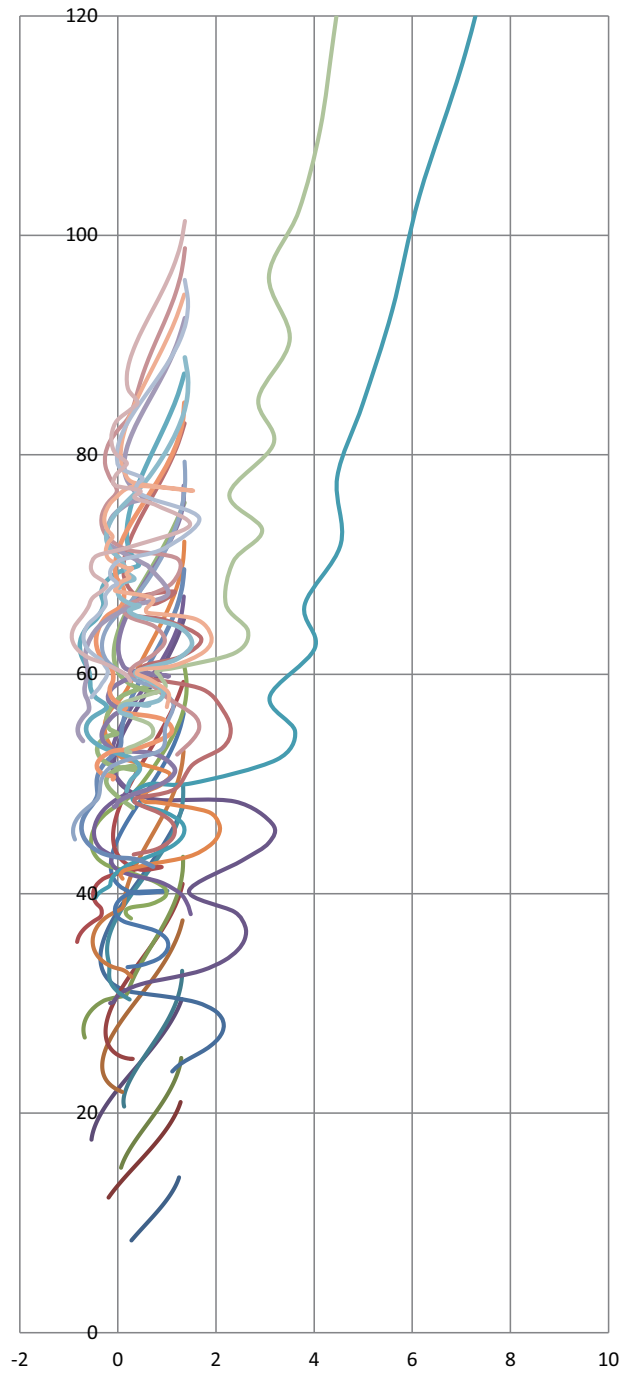


FIGURE 9. The behavior of  $s_1$  of the curves described in Figure 7.

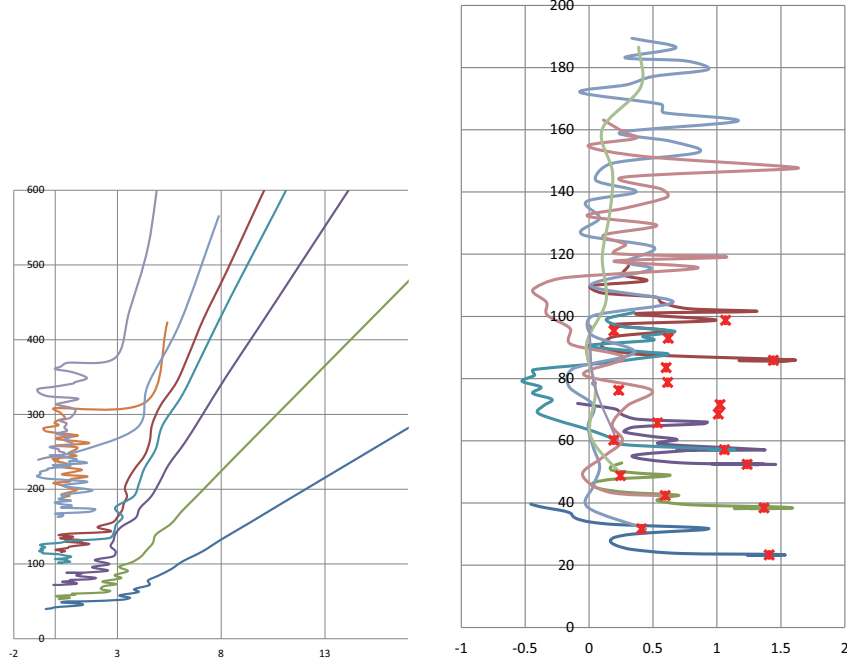


FIGURE 10. The behavior of  $s_1$  (left) and  $s_2$  (right) of curves described in Figure 8. The cross marks on the right figure are solutions of  $\zeta(s_2) = 1$ . There may be more solutions, but we only mark the solutions we checked.

and  $|s_2^{(m)}|$  remains bounded as  $m \rightarrow \infty$ , then  $s_2^{(m)}$  tends to a solution of  $\zeta(s_2) = 1$ .

(ii) Conversely, for any solution  $\rho_2$  of  $\zeta(\rho_2) = 1$  and any  $\varepsilon > 0$ , we can find a zero of  $\zeta_2(s_1, s_2)$  such that  $|s_2 - \rho_2| < \varepsilon$ .

*Proof.* (i) Recall the harmonic product formula

$$(4.1) \quad \zeta(s_1)\zeta(s_2) = \zeta_2(s_1, s_2) + \zeta_2(s_2, s_1) + \zeta(s_1 + s_2).$$

Use this formula with  $s_1 = s_1^{(m)}$  and  $s_2 = s_2^{(m)}$ . Then  $\zeta_2(s_1^{(m)}, s_2^{(m)}) = 0$ . Also, when  $m \rightarrow \infty$ , by the assumptions of (i) we have  $\zeta(s_1^{(m)}) \rightarrow 1$  and  $\zeta(s_1^{(m)} + s_2^{(m)}) \rightarrow 1$ . Further we have

$$(4.2) \quad \zeta_2(s_2^{(m)}, s_1^{(m)}) = \sum_{n_1=1}^{\infty} \sum_{n_2=1}^{\infty} \frac{1}{n_1^{s_2^{(m)}} (n_1 + n_2)^{s_1^{(m)}}} \rightarrow 0$$

because  $n_1 + n_2 \geq 2$ . Therefore taking  $m \rightarrow \infty$  in (4.1) we find that  $\zeta(s_2^{(m)}) \rightarrow 1$ .

(ii) Choose  $\varepsilon_0 > 0$  sufficiently small so that  $\zeta(s_2)$  is holomorphic in the region  $|s_2 - \rho_2| < \varepsilon_0$ , and the only solution of  $\zeta(s_2) = 1$  in this region is  $\rho_2$ . Let  $0 < \varepsilon < \varepsilon_0$ , and define

$$M(\varepsilon) = \max_{|s_2 - \rho_2| = \varepsilon} |\zeta(s_2)|, \quad m(\varepsilon) = \min_{|s_2 - \rho_2| = \varepsilon} |\zeta(s_2) - 1|.$$

Note that  $m(\varepsilon) > 0$  because of the choice of  $\varepsilon_0$ . Choose real numbers  $R_1(\varepsilon)$ ,  $R_2(\varepsilon)$ , and  $R_3(\varepsilon)$  satisfying

$$(4.3) \quad |\zeta(s_1) - 1| < \frac{m(\varepsilon)}{3M(\varepsilon)}$$

if  $\Re s_1 > R_1(\varepsilon)$ ,

$$(4.4) \quad |\zeta(s_1 + s_2) - 1| < \frac{1}{3}m(\varepsilon)$$

if  $\Re s_1 > R_2(\varepsilon)$  and  $|s_2 - \rho_2| \leq \varepsilon$ , and

$$(4.5) \quad |\zeta_2(s_2, s_1)| < \frac{1}{3}m(\varepsilon)$$

if  $\Re s_1 > R_3(\varepsilon)$  and  $|s_2 - \rho_2| \leq \varepsilon$  (where the reasoning of (4.5) is similar to that of (4.2)). Put  $R(\varepsilon) = \max\{R_1(\varepsilon), R_2(\varepsilon), R_3(\varepsilon)\}$ . Since from (4.1) we have

$$\zeta_2(s_1, s_2) - (\zeta(s_2) - 1) = \zeta(s_2)(\zeta(s_1) - 1) - (\zeta(s_1 + s_2) - 1) - \zeta_2(s_2, s_1),$$

using (4.3), (4.4) and (4.5) we find

$$(4.6) \quad |\zeta_2(s_1, s_2) - (\zeta(s_2) - 1)| < M(\varepsilon) \cdot \frac{m(\varepsilon)}{3M(\varepsilon)} + \frac{1}{3}m(\varepsilon) + \frac{1}{3}m(\varepsilon) = m(\varepsilon)$$

if  $\Re s_1 > R(\varepsilon)$  and  $|s_2 - \rho_2| \leq \varepsilon$ . Now fix an  $s_1$  satisfying  $\Re s_1 > R(\varepsilon)$ . From (4.6) we have

$$|\zeta_2(s_1, s_2) - (\zeta(s_2) - 1)| < |\zeta(s_2) - 1|$$

on the circle  $|s_2 - \rho_2| = \varepsilon$ . Therefore by Rouché's theorem we have that the number of zeros (with multiplicity) of  $\zeta_2(s_1, s_2)$  (as a function in  $s_2$ ) in the region  $|s_2 - \rho_2| < \varepsilon$  is equal to the number of zeros of  $\zeta(s_2) - 1$  in the same region, but the latter is 1. This implies the assertion of (ii).  $\square$

*Remark 3.* The assertion (i) of the above proposition is, if  $\Re s_2 > 1$ , obvious. Because in this case we can use (1.1). Letting  $\Re s_1 \rightarrow \infty$  on (1.1), we see that the right-hand side tends to  $\sum_{n_2=1}^{\infty} (1 + n_2)^{-s_2} = \zeta(s_2) - 1$ .

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